FP3 June 14 IAL M.A. Xprime 2

Leave blank

1. Given that $y = \arctan\left(\frac{2x}{3}\right)$,

(a) find $\frac{dy}{dx}$, giving your answer in its simplest form.

(2)

(b) Use integration by parts to find

$$\int \arctan\left(\frac{2x}{3}\right) dx$$

(4)

1(a) $y = \arctan\left(\frac{2x}{3}\right)$

$\therefore \tan y = \frac{2x}{3}$

$\frac{s^2}{c^2} + \frac{c^2}{c^2} = \frac{1}{c^2}$

$\therefore \sec^2 y \frac{dy}{dx} = \frac{2}{3}$

$\therefore (\tan^2 y + 1)\frac{dy}{dx} = \frac{2}{3}$

$\therefore \left(\frac{4x^2}{9} + 1\right)\frac{dy}{dx} = \frac{2}{3}$

$\therefore \frac{4x^2+9}{9}\left(\frac{dy}{dx}\right) = \frac{2}{3}$

$\therefore \frac{dy}{dx} = \frac{6}{4x^2+9}$

Question 1 continued

Let $u = \arctan\left(\frac{2x}{3}\right)$ $\quad u' = \frac{6}{4x^2+9}$

$v' = 1 \quad v = x$

$$\int \arctan \frac{2x}{3} = x\arctan\left(\frac{2x}{3}\right) - \int \frac{6x}{4x^2+9}\,dx$$

$$= x\arctan\left(\frac{2x}{3}\right) - \frac{3}{4}\int \frac{8x}{4x^2+9}\,dx$$

$$= x\arctan\frac{2x}{3} - \frac{3}{4}\ln(4x^2+9) + C$$

Leave blank

2. The line with equation $x = 9$ is a directrix of an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{8} = 1$$

where a is a positive constant.

Find the two possible exact values of the constant a.

(6)

2. Directrix

$x = 9 \Rightarrow 9 = \frac{a}{e}$ $\therefore e = \frac{a}{9}$

Eccentricity: $8 = a^2\left(1 - \frac{a^2}{81}\right)$

$\therefore 8 = a^2 - \frac{a^4}{81}$

$\therefore \frac{a^4}{81} - a^2 + 8 = 0$ $\quad a^4 - 81a^2 + 648 = 0$

$\cancel{\therefore (a^2 - 9)(8a^2 - 9) = 0}$

$\therefore (a^2 - 72)(a^2 - 9) = 0$

$a > 0$

$\Rightarrow a^2 = 72 \Rightarrow a = 6\sqrt{2}$

$a^2 = 9 \Rightarrow a = 3$

blank

3. Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials,

(a) prove that

$$\cosh^2 x - \sinh^2 x \equiv 1$$

(2)

(b) find algebraically the exact solutions of the equation

$$2 \sinh x + 7 \cosh x = 9$$

giving your answers as natural logarithms.

(5)

(a)

$$LHS = \cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x)$$

$$= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)$$

$$\cancel{= e^x \times} = \left(\frac{e^x + e^x + e^{-x} - e^{-x}}{2}\right)\left(\frac{e^x - e^x + e^{-x} + e^{-x}}{2}\right)$$

$$= e^x \times e^{-x} = 1 = RHS$$

as required.

(b) $2\sinh x + 7\cosh x = 9$

$\therefore \quad e^x - e^{-x} + \frac{7}{2}e^x + \frac{7}{2}e^{-x} = 9$

$\therefore \quad \frac{9}{2}e^x + \frac{5}{2}e^{-x} = 9$

$\therefore \quad 9e^x + 5e^{-x} = 18$

blank

Question 3 continued

$\times e^{x}$

$\therefore \quad 9e^{2x} - 18e^{x} + 5 = 0$

$\therefore \quad (3e^{x} - 5)(3e^{x} - 1) = 0$

$\therefore \quad e^{x} = \frac{5}{3} \Rightarrow x = \ln\frac{5}{3}$

$e^{x} = \frac{1}{3} \Rightarrow x = \ln\frac{1}{3}$

blank

4. A non-singular matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 3 & k & 0 \\ k & 2 & 0 \\ k & 0 & 1 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

(a) Find, in terms of k, the inverse of the matrix $\mathbf{M}$.

(5)

The point A is mapped onto the point $(-5, 10, 7)$ by the transformation represented by the matrix

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(b) Find the coordinates of the point A.

(3)

4(a). $\det(M) = 3(2) - k(k) = 6 - k^2$

$$C = \begin{pmatrix} +(2) & -(k) & +(-2k) \\ -(k) & +(3) & -(-k^2) \\ +(0) & -(0) & +(6-k^2) \end{pmatrix}$$

$$\therefore C^T = \begin{pmatrix} 2 & -k & 0 \\ -k & 3 & 0 \\ -2k & k^2 & 6-k^2 \end{pmatrix}$$

$$\therefore M^{-1} = \frac{1}{6-k^2} \begin{pmatrix} 2 & -k & 0 \\ -k & 3 & 0 \\ -2k & k^2 & 6-k^2 \end{pmatrix}$$

Question 4 continued

(b). $K = 1$

$$\therefore M^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ -2 & 1 & 5 \end{pmatrix}$$

$$MA = \begin{pmatrix} 5 \\ 10 \\ 7 \end{pmatrix}$$

$$\therefore M^{-1}MA = M^{-1}\begin{pmatrix} -5 \\ 10 \\ 7 \end{pmatrix}$$

$$\therefore A = \frac{1}{5}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} -5 \\ 10 \\ 7 \end{pmatrix}$$

$$\therefore A = \frac{1}{5}\begin{pmatrix} -20 \\ 35 \\ 55 \end{pmatrix}$$

$$\therefore A: (-4, 7, 11)$$

Leave blank

5. Given that

$$I_n = \int_0^{\frac{\pi}{4}} \cos^n \theta \, d\theta, \qquad n \geqslant 0$$

(a) prove that, for $n \geqslant 2$,

$$nI_n = \left(\frac{1}{\sqrt{2}}\right)^n + (n-1)I_{n-2}$$

(6)

(b) Hence find the exact value of I_5, showing each step of your working.

(5)

Let $\theta = x$

5(a). $I_n = \int_0^{\pi/4} \cos^n x \, dx = \int_0^{\pi/4} \cos^{n-2} x \cos^2 x \, dx$

~~Let $u = \cos^{n-2} x$ $\quad u' = -\sin x(n-2)\cos^{n-3} x$~~

~~$v' = \cos^2 x$ $\quad \therefore v =$ $\quad$ Let $\theta =$~~

~~$= \frac{\cos 2x + 1}{2}$~~

$c^2 - s^2$

$c^2 - (1 - c^2)$

$2c^2 - 1$

Let $u = \cos^{n-1} x$ $\quad u' = -\sin x(n-1)\cos^{n-2} x$

$v' = \cos x$ $\quad v = \sin x$

$\therefore I_n = \left[\sin x \cos^{n-1} x\right]_0^{\pi/4} + (n-1)\int_0^{\pi/4} \sin^2 x \cos^{n-2} x$

use $\sin^2 = 1 - \cos^2 x$

$\therefore I_n = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^{n-1} + (n-1)\int_0^{\pi/4} \cos^{n-2} x - \cos^n x \, dx$

$\therefore I_n = \left(\frac{1}{\sqrt{2}}\right)^n + (n-1)\left(I_{n-2} - I_n\right)$

Question 5 continued

$\therefore \quad I_n = \left(\frac{1}{\sqrt{2}}\right)^n + (n-1) I_{n-2} - (n-1) I_n$

$\therefore \quad n I_n = \left(\frac{1}{\sqrt{2}}\right)^n + (n-1) I_{n-2}$

// as required.

(b) ~~$I_5 =$~~ let $n=5$

$5 I_5 = \left(\frac{1}{\sqrt{2}}\right)^5 + 4 I_3$

~~4~~ $3 I_3 = \left(\frac{1}{\sqrt{2}}\right)^3 + 2 I_1$

~~$I_1 = \int_0^{\pi/4} \cos x \, dx = [\sin x]_0^{\pi/4} = \frac{1}{\sqrt{2}}$~~

$I_1 = \int_0^{\pi/4} \cos x \, dx = [\sin x]_0^{\pi/4} = \frac{1}{\sqrt{2}}$

$\therefore \quad I_3 = \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)$

$\therefore \quad 5 I_5 = \left(\frac{1}{\sqrt{2}}\right)^5 + \frac{4}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{8}{3}\left(\frac{1}{\sqrt{2}}\right)$

Question 5 continued

$$\therefore 5I_5 = \frac{\sqrt{2}}{8} + \frac{4}{3}\left(\frac{\sqrt{2}}{4}\right) + \frac{8}{3\sqrt{2}}$$

$$\therefore 5I_5 = \frac{43\sqrt{2}}{24}$$

$$\therefore I_5 = \frac{43}{120}\sqrt{2}$$

0-506

blank

6. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{4} = 1$$

The line l is a tangent to H at the point $P\ (4\cosh\alpha,\ 2\sinh\alpha)$, where α is a constant, $\alpha \neq 0$

(a) Using calculus, show that an equation for l is

$$2y\sinh\alpha - x\cosh\alpha + 4 = 0$$

(4)

The line l cuts the y-axis at the point A.

(b) Find the coordinates of A in terms of α.

(2)

The point B has coordinates $(0,\ 10\sinh\alpha)$ and the point S is the focus of H for which $x > 0$

(c) Show that the line segment AS is perpendicular to the line segment BS.

(5)

6(a) @ P, $\frac{dy}{dx} = \frac{dy/d\alpha}{dx/d\alpha} = \frac{2\cosh\alpha}{4\sinh\alpha} = \frac{\cosh\alpha}{2\sinh\alpha}$

$\therefore\ y - y_1 = m(x - x_1)$

$\Rightarrow\ y - 2\sinh\alpha = \frac{\cosh\alpha}{2\sinh\alpha}(x - 4\cosh\alpha)$

$\therefore\ y - 2\sinh\alpha = \frac{\cosh\alpha}{2\sinh\alpha}x - \frac{2\cosh^2\alpha}{\sinh\alpha}$

$\therefore$ (x 2sinhα) $\therefore\ 2y\sinh\alpha - 4\sinh^2\alpha = x\cosh\alpha - 4\cosh^2\alpha$

$\therefore\ 2y\sinh\alpha - x\cosh\alpha + 4(\cosh^2\alpha - \sinh^2\alpha) = 0$

$\Rightarrow\ 2y\sinh\alpha - x\cosh\alpha + 4 = 0$

// as required.

Question 6 continued

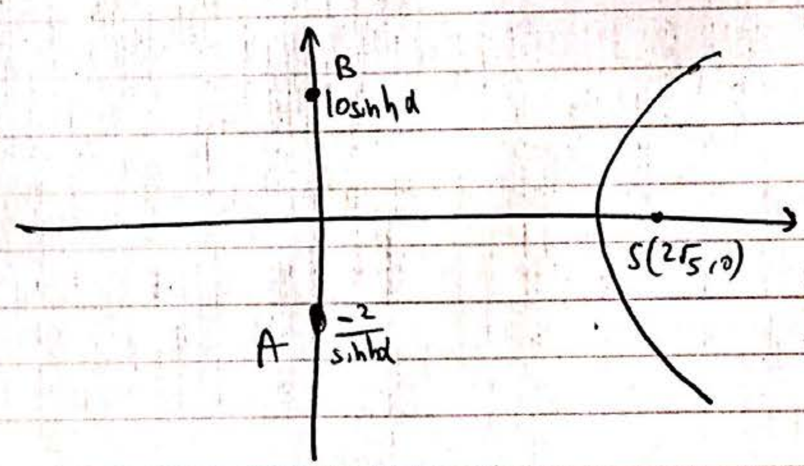
(b) @ A, $x=0$

$\therefore \quad 2y\sinh\alpha = -4$

$\therefore \quad y = \dfrac{-2}{\sinh\alpha}$

$\therefore \quad A: \left(0, \ -\dfrac{2}{\sinh\alpha}\right)$

(c)



Gradient $AS = \dfrac{\frac{2}{\sinh\alpha}}{2\sqrt{5}} = \dfrac{1}{\sqrt{5}\sinh\alpha}$

Gradient $BS = \dfrac{-10\sinh\alpha}{2\sqrt{5}} = -\dfrac{5}{\sqrt{5}}\sinh\alpha = -\sqrt{5}\sinh\alpha$

$\therefore$ Product of gradients $= \dfrac{1}{\sqrt{5}\sinh\alpha} \times -\sqrt{5}\sinh\alpha = -1$

$\therefore$ Product of gradiants $= -1 \quad \therefore$ BS & AS are perpendicular.

Leave blank

7. The curve C has parametric equations

$$x = 3t^2, \quad y = 12t, \quad 0 \leqslant t \leqslant 4$$

The curve C is rotated through 2π radians about the x-axis.

(a) Show that the area of the surface generated is

$$\pi(a\sqrt{5} + b)$$

where a and b are constants to be found.

(6)

(b) Show that the length of the curve C is given by

$$k\int_0^4 \sqrt{(t^2 + 4)}\,\mathrm{d}t$$

where k is a constant to be found.

(1)

(c) Use the substitution $t = 2\sinh\theta$ to show that the exact value of the length of the curve C is

$$24\sqrt{5} + 12\ln(2 + \sqrt{5})$$

(6)

7(a). $x = 3t^2 \qquad y = 12t$

$\frac{\partial x}{\partial t} = 6t \qquad \frac{\partial y}{\partial t} = 12$

$\therefore$

$S = 2\pi\int_0^4 12t\sqrt{36t^2 + 144}\, dt = \frac{1}{3}\pi\int_0^4 72t\,(36t^2+144)^{1/2}\, dt$

$= \frac{\pi}{3}\left[\frac{2}{3}(36t^2+144)^{3/2}\right]_0^4$

$= \frac{\pi}{3}\left(\frac{2}{3}(720)^{3/2} - 1152\right)$

$= \frac{2\pi}{9}(12\sqrt{5})^3 - 384\pi$

Question 7 continued

$$= \frac{2\pi}{9} \times 1728 \times 5\sqrt{5} \quad - 384\pi$$

$$= 1920\sqrt{5}\,\pi \quad - 384\pi$$

$$= \pi\left(1920\sqrt{5} - 384\right)$$

1 12281

(c) $S = \int_0^4 \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2} \; \partial t$

$$\therefore S = \int_0^4 \sqrt{36t^2 + 144} \; \partial t$$

$$= \int_0^4 \sqrt{36}\sqrt{t^2+4} \; \partial t$$

$$\therefore S = 6\int_0^4 \sqrt{t^2+4} \; \partial t \qquad k = 6$$

as required

(c) $t = 2\sinh\theta \Rightarrow \frac{\partial t}{\partial \theta} = 2\cosh\theta = 2\sqrt{1+\sinh^2\theta}$

$= 2\cancel{\sqrt{1+\frac{t^2}{4}}}$

$-s^2 = 1$

$\therefore \cancel{\partial t = 2\sqrt{1+\frac{t^2}{4}}\;\partial\theta} \;\therefore \partial t = 2\cosh\theta\,\partial\theta$

Leave blank

Question 7 continued

$$\sqrt{t^2+4} \equiv \sqrt{(2\sinh\theta)^2+4}$$

$$= \sqrt{4\sinh^2\theta+4}$$

$$= \sqrt{4(1+\sinh^2\theta)} = 2\cosh\theta$$

~~Now~~ Consider new limits:

$t = 4 \Rightarrow 2\sinh\theta = 4 \Rightarrow \theta = \text{arsinh}\,2$

$t=0 \Rightarrow 2\sinh\theta = 0 \Rightarrow \theta = \text{arsinh}\,0 = 0$

Now:

$\sqrt{t^2+4} = 2\cosh\theta$ & $dt = 2\cosh\theta\, d\theta$

$$\therefore 6\int_0^4 \sqrt{t^2+4}\, dt = 6\int_0^{\text{arsinh}2} 2\cosh\theta \cdot 2\cosh\theta\, d\theta$$

$c^2-s^2=1$

$$= 24\int_0^{\text{arsinh}2} \cosh^2\theta\, d\theta$$

$c^2+s^2=\cosh 2$

c^2+c^2-1

$2c^2-1$

$$= 12\int_0^{\text{arsinh}2} \cosh 2\theta + 1\, d\theta$$

$$= 12\left[\tfrac{1}{2}\sinh 2\theta + \theta\right]_0^{\text{arsinh}2 = \ln(2+\sqrt{5})}$$

Question 7 continued

$$= 12\left(\tfrac{1}{2}\sinh \ln[(2+\sqrt{5})^2] \quad + \ln(2+\sqrt{5})\right)$$

$$= 12\left(\tfrac{1}{4}\left(e^{\ln(2+\sqrt{5})^2} - e^{-\ln(2+\sqrt{5})^2}\right) + \ln(2+\sqrt{5})\right)$$

$$= 12\left(\tfrac{1}{4}\left(9+4\sqrt{5} - (9-4\sqrt{5})\right) + \ln(2+\sqrt{5})\right)$$

$$= 3(8\sqrt{5}) + 12\ln(2+\sqrt{5})$$

$$= 24\sqrt{5} + 12\ln(2+\sqrt{5})$$

// as required.

Q7

(Total 13 marks)

Leave blank

8. The line l has equation

$$\mathbf{r} = (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \lambda(3\mathbf{i} + 2\mathbf{j} + \mathbf{k}), \text{ where } \lambda \text{ is a scalar parameter,}$$

and the plane Π has equation

$$\mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 19$$

(a) Find the coordinates of the point of intersection of l and Π.

(4)

The perpendicular to Π from the point A (2, 1, –2) meets Π at the point B.

(b) Verify that the coordinates of B are (4, 3, –6).

(3)

The point A (2, 1, –2) is reflected in the plane Π to give the image point A'.

(c) Find the coordinates of the point A'.

(2)

(d) Find an equation for the line obtained by reflecting the line l in the plane Π, giving your answer in the form

$$\mathbf{r} \times \mathbf{a} = \mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors to be found.

(4)

(a) $\Pi: \quad x + y - 2z = 19$

$l: \quad \underset{\sim}{r} = \begin{pmatrix} 2+3\lambda \\ 1+2\lambda \\ -2+\lambda \end{pmatrix}$

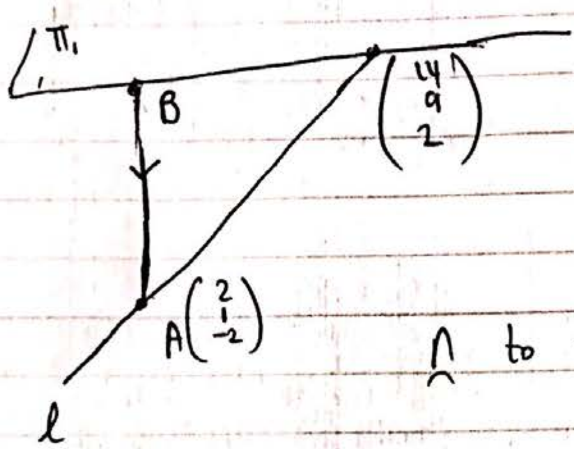
$\therefore \quad 2+3\lambda + 1 + 2\lambda - 2(-2+\lambda) = 19$

$\therefore \quad 3\lambda + 7 = 19 \Rightarrow \lambda = 4$

$\therefore$ @ intersection: $\underset{\sim}{r} = \begin{pmatrix} 14 \\ 9 \\ 2 \end{pmatrix}$ //

Question 8 continued

(b)



$A: \begin{pmatrix}2\\1\\-2\end{pmatrix}$ lies on l

$\underset{\sim}{n}$ to $\Pi \Rightarrow \underset{\sim}{n} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$

* $\underset{\sim}{n}$ is in direction $\overrightarrow{BA}$! we need $\overrightarrow{AB}$

$\therefore$ $\cancel{\overrightarrow{AB}}$

AB has eqn : $\underset{\sim}{r} = \begin{pmatrix}2\\1\\-2\end{pmatrix} + \mu\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

$= \begin{pmatrix}2-\mu\\1-\mu\\-2+2\mu\end{pmatrix}$

line AB intersects Π_1

$\therefore \quad x + y - 2z = 19$

$\Rightarrow \quad 2-\mu + 1-\mu - 2(-2+2\mu) = 19$

$\therefore \quad 7 - 6\mu = 19$

$\therefore \quad \mu = -2$ @ intersection

$\Rightarrow$ @ intersection $\underset{\sim}{r} = \begin{pmatrix}2 - -2\\1 - -2\\-2+2(-2)\end{pmatrix} = \begin{pmatrix}4\\3\\-6\end{pmatrix}$

$\therefore$ B has coords $\begin{pmatrix}4\\3\\-6\end{pmatrix}$ // as required.

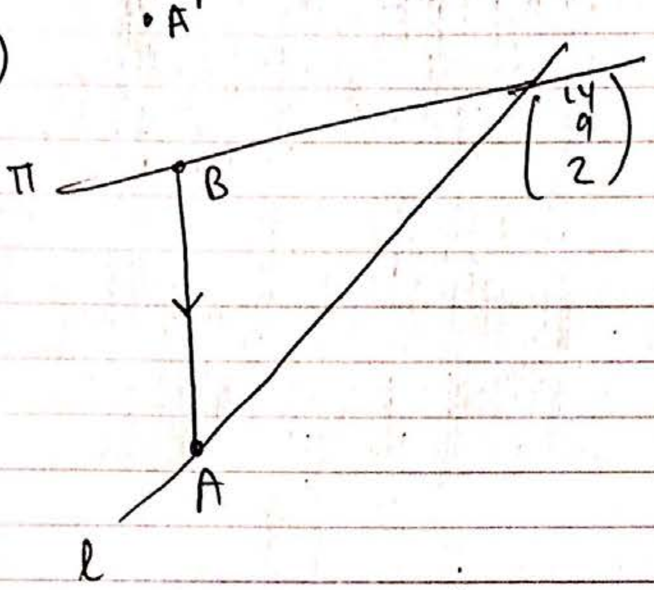
Scanned by CamScanner

Turn over

Leave blank

Question 8 continued

(C)



$OA' = OA + 2AB$

$\therefore A' = \begin{pmatrix}2\\1\\-2\end{pmatrix} + 2\left(\begin{pmatrix}4\\3\\-6\end{pmatrix} - \begin{pmatrix}2\\1\\-2\end{pmatrix}\right)$

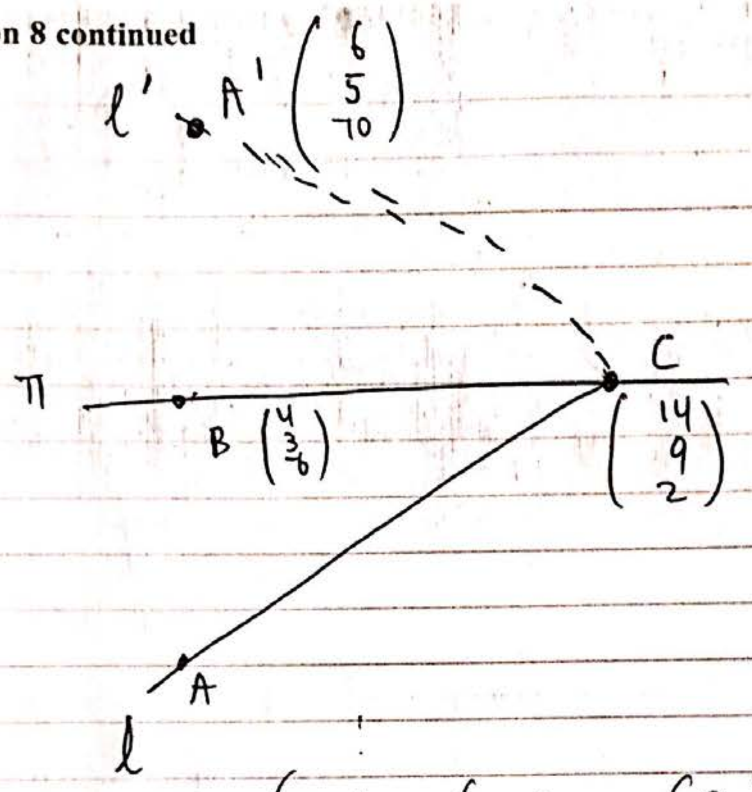
$= \begin{pmatrix}2\\1\\-2\end{pmatrix} + 2\begin{pmatrix}2\\2\\-4\end{pmatrix}$

$A' = \begin{pmatrix}6\\5\\-10\end{pmatrix}$ //

~~(d)~~

Question 8 continued

(d)



Let $C = \begin{pmatrix}14\\9\\2\end{pmatrix}$

$$\overrightarrow{A'C} = \begin{pmatrix}14\\9\\2\end{pmatrix} - \begin{pmatrix}6\\5\\-10\end{pmatrix} = \begin{pmatrix}8\\4\\12\end{pmatrix}$$

$\therefore$ l' has eqn $\underset{\sim}{r} = \begin{pmatrix}6\\5\\-10\end{pmatrix} + t\begin{pmatrix}8\\4\\12\end{pmatrix}$

$(\underset{\sim}{r} - \underset{\sim}{a}) \times \underset{\sim}{b} = 0$

$\begin{matrix}6 & 5 & -10 & 6 & 5 & -10\\ 8 & 4 & 12 & 8 & 4 & 12\end{matrix}$

$\underset{\sim}{r} \times \underset{\sim}{b} = \underset{\sim}{a} \times \underset{\sim}{b}$

$$\therefore \quad \underset{\sim}{r} \times \begin{pmatrix}8\\4\\12\end{pmatrix} = \begin{pmatrix}6\\5\\-10\end{pmatrix} \times \begin{pmatrix}8\\4\\12\end{pmatrix} = \begin{pmatrix}100\\-152\\-16\end{pmatrix}$$

$$\therefore \quad \underset{\sim}{r} \times \cancel{4}\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}25\\-38\\-4\end{pmatrix} \Rightarrow \underset{\sim}{r} \times \begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}25\\-38\\-4\end{pmatrix}$$